Sampling recovery. Lecture 1. Recovery in the $L_p$ norms.

Vladimir Temlyakov

May, 2021
Functions of the form

\[ t(x) = \sum_{|k| \leq n} c_k e^{ikx} = a_0/2 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \]

are called trigonometric polynomials of order \( n \). The set of such polynomials we denote by \( \mathcal{T}(n) \).
Trigonometric polynomials. Dirichlet kernel.

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are called trigonometric polynomials of order \( n \). The set of such polynomials we denote by \( \mathcal{T}(n) \).

The Dirichlet kernel of order \( n \)

\[ \mathcal{D}_n(x) := \sum_{|k| \leq n} e^{ikx} = e^{-inx} \frac{e^{i(2n+1)x} - 1}{e^ix - 1} \frac{1}{e^{ix} - 1} \]

\[ = (\sin(n + 1/2)x) / \sin(x/2). \]
Denote
\[ x^j := \frac{2\pi j}{2n + 1}, \quad j = 0, 1, \ldots, 2n. \]
Clearly, the points \( x^j, j = 1, \ldots, 2n, \) are zeros of the Dirichlet kernel \( D_n \) on \([0, 2\pi]\).
Denote

\[ x^j := \frac{2\pi j}{2n + 1}, \quad j = 0, 1, \ldots, 2n. \]

Clearly, the points \( x^j, j = 1, \ldots, 2n \), are zeros of the Dirichlet kernel \( D_n \) on \([0, 2\pi]\). Consequently, for any continuous \( f \)

\[
I_n(f)(x) := (2n + 1)^{-1} \sum_{j=0}^{2n} f(x^j) D_n(x - x^j)
\]

interpolates \( f \) at points \( x^j \): \( I_n(f)(x^j) = f(x^j), \ j = 0, 1, \ldots, 2n. \)
It is easy to check that for any $t \in \mathcal{T}(n)$ we have $l_n(t) = t$. Using this and the inequality

$$|D_n(x)| \leq \min(2n + 1, \pi/|x|), \quad |x| \leq \pi,$$

we obtain

$$\|f - l_n(f)\|_\infty \leq C \ln(n + 1)E_n(f)_\infty,$$

where $E_n(f)_p$ is the best approximation of $f$ in the $L_p$ norm by polynomials from $\mathcal{T}(n)$. 
The de la Vallée Poussin kernels

\[ V_{2n}(x) := n^{-1} \sum_{k=n}^{2n-1} D_k(x) = \frac{\cos nx - \cos 2nx}{n(\sin(x/2))^2}. \]
The de la Vallée Poussin kernels $V_n$ are even trigonometric polynomials of order $2n - 1$ with the majorant

$$|V_n(x)| \leq C \min(n, 1/(nx^2)), \ |x| \leq \pi.$$
The de la Vallée Poussin kernels

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\[ |V_n(x)| \leq C \min(n, 1/(nx^2)), \ |x| \leq \pi. \]

Consider the following recovery operator

\[ R_n(f) := (4n)^{-1} \sum_{j=1}^{4n} f(x(j)) V_n(x - x(j)), \quad x(j) := \pi j/(2n). \]
It is easy to check that for any \( t \in \mathcal{T}(n) \) we have \( R_n(t) = t \). Using this and the above majorant we obtain

\[
\| f - R_n(f) \|_\infty \leq C E_n(f)_\infty.
\]
Properties of $R_n(f)$

It is easy to check that for any $t \in T(n)$ we have $R_n(t) = t$. Using this and the above majorant we obtain

$$\|f - R_n(f)\|_\infty \leq C E_n(f)_\infty.$$ 

What about error in the $L_p$, $p \in [1, \infty)$? Let $\varepsilon := \{\varepsilon_k\}_{k=0}^\infty$ be a non-increasing sequence of non-negative numbers. Define

$$E(\varepsilon, p) := \{f \in C : E_k(f)_p \leq \varepsilon_k, \ k = 0, 1, \ldots \}.$$
Theorem (VT, 1985)

Assume that a sequence \( \varepsilon \) satisfies the conditions: for all \( s = 0, 1, \ldots \), we have

\[
\sum_{\nu=s+1}^{\infty} \varepsilon_{2\nu} \leq B\varepsilon_{2s}, \quad \varepsilon_s \leq D\varepsilon_{2s}.
\]

Then for \( p \in [1, \infty) \)

\[
\sup_{f \in E(\varepsilon, p)} \| f - R_n(f) \|_p \asymp \sum_{\nu=0}^{\infty} 2^{\nu/p} \varepsilon_{n2\nu}.
\]
Theorem (VT, 1985)

Assume that a sequence $\varepsilon$ satisfies the conditions: for all $s = 0, 1, \ldots$ we have

$$\sum_{\nu=s+1}^{\infty} \varepsilon_{2\nu} \leq B\varepsilon_{2s}, \quad \varepsilon_s \leq D\varepsilon_{2s}.$$ 

Then for $p \in [1, \infty)$

$$\sup_{f \in E(\varepsilon, p)} \|f - R_n(f)\|_p \preceq \sum_{\nu=0}^{\infty} 2^{\nu/p} \varepsilon_n 2^\nu.$$ 

This theorem for $1 \leq p \leq 2$ was proved in VT, 1985. A similar proof works for other $p$. 
Norms of operators

In the case of space $C$ ($p = \infty$) we have

$$\|R_n\|_{C \to C} \leq C.$$ 

This allows us to obtain the inequality $\|f - R_n(f)\|_{\infty} \leq C E_n(f)_{\infty}$. 

Operators $R_n$ are not defined on $L^p$, when $p < \infty$. What to do?

Historically, the first idea was to consider the operator $R_n$ $J_r$ where

$$J_r(f)(x) := (2\pi)^{-1} \int_T f(x-y) F_r(y) \, dy,$$

$F_r(y) := 1 + \sum_{k=1}^{\infty} k^{-r} \cos(ky - r\pi/2)$.

It was proved in VT, 1985 that for $r > 1/p$ we have ($I$ is the identity operator)

$$\|I - R_n J_r\|_{L^p \to L^p} \leq C(r, p).$$
Norms of operators

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This allows us to obtain the inequality $\|f - R_n(f)\|_{\infty} \leq C E_n(f)_{\infty}$. Operators $R_n$ are not defined on $L_p$, when $p < \infty$. What to do? Historically, the first idea was to consider the operator $R_nJ_r$ where

$$J_r(f)(x) := (2\pi)^{-1} \int_{\mathbb{T}} f(x - y)F_r(y)dy,$$

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It was proved in **VT, 1985** that for $r > 1/p$ we have ($I$ is the identity operator)

$$\|I - R_nJ_r\|_{L_p \to L_p} \leq C(r, p)n^{-r}.$$
The following inequalities turns out to be more convenient. Denote

\[ V_s(f)(x) := (2\pi)^{-1} \int_{\mathbb{T}} f(x - y)V_s(y)dy. \]

Then (VT, 1993) we have for \( s \geq n \)

\[ \|R_nV_s\|_{L^p \to L^p} \leq C(s/n)^{1/p}, \quad 1 \leq p \leq \infty \]

and

\[ \|I_nV_s\|_{L^p \to L^p} \leq C(p)(s/n)^{1/p}, \quad 1 < p < \infty. \]
For a fixed $m$ and a set of points $\xi := \{\xi^j\}_{j=1}^m \subset \Omega$, let $\Phi_\xi$ be a linear operator from $C^m$ into $L_p(\Omega, \mu)$. Denote for a class $F$ (usually, centrally symmetric and compact subset of $L_p(\Omega, \mu)$)

$$\varrho_m(F, L_p) := \inf_{\text{linear } \Phi_\xi; \xi \in F} \sup_{f \in F} \|f - \Phi_\xi(f(\xi^1), \ldots, f(\xi^m))\|_p.$$ 

The above described recovery procedure is a linear procedure.
For a fixed $m$ and a set of points $\xi := \{\xi^j\}_{j=1}^m \subset \Omega$, let $\Phi_\xi$ be a linear operator from $\mathbb{C}^m$ into $L_p(\Omega, \mu)$. Denote for a class $F$ (usually, centrally symmetric and compact subset of $L_p(\Omega, \mu)$)

$$\rho_m(F, L_p) := \inf_{\text{linear } \Phi_\xi; \xi \in F} \sup_{f \in F} \|f - \Phi_\xi(f(\xi^1), \ldots, f(\xi^m))\|_p.$$ 

The above described recovery procedure is a linear procedure. The following modification of the above recovery procedure is also of interest. We now allow any mapping $\Phi_\xi : \mathbb{C}^m \to X_N \subset L_p(\Omega, \mu)$ where $X_N$ is a linear subspace of dimension $N \leq m$ and define

$$\rho^*_m(F, L_p) := \inf_{\Phi_\xi; \xi \in F; X_N, N \leq m} \sup_{f \in F} \|f - \Phi_\xi(f(\xi^1), \ldots, f(\xi^m))\|_p.$$ 

In both of the above cases we build an approximant, which comes from a linear subspace of dimension at most $m$. 

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Univariate smoothness classes

Define

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**Theorem (VT, 1993)**

Let \( 1 \leq q, p \leq \infty \) and \( r > 1/q \). Then

\[ \varrho_{4m}(W^r_q, L_p) \asymp \sup_{f \in W^r_q} \| f - R_m(f) \|_p \asymp m^{-r+(1/q-1/p)+}. \]

*In the case \( 1 < p < \infty \) the above estimates are valid for the operator \( I_m \) instead of the operator \( R_m \).*
For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}_+^d$ define

$$J_{\mathbf{r}}(f)(\mathbf{x}) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(\mathbf{x} - \mathbf{y}) F_{\mathbf{r}}(\mathbf{y}) d\mathbf{y},$$

$$F_{\mathbf{r}}(\mathbf{y}) := \prod_{j=1}^d F_{r_j}(y_j)$$
For $r = (r_1, \ldots, r_d) \in \mathbb{R}_+^d$ define

$$J_r(f)(x) := (2\pi)^{-d} \int_{T^d} f(x - y) F_r(y) \, dy,$$

$$F_r(y) := \prod_{j=1}^{d} F_{r_j}(y_j)$$

and

$$W_q^r := \{ f : f = J_r(\varphi), \| \varphi \|_q \leq 1 \}.$$
Let for $i = 1, \ldots, d$ operator $R_n^i$ be the operator $R_n$ acting with respect to the variable $x_i$. Denote

$$\Delta_s^i := R_{2s}^i - R_{2s-1}^i, \quad R_{1/2} = 0,$$

and for $s = (s_1, \ldots, s_d) \in \mathbb{N}_0^d$

$$\Delta_s := \prod_{i=1}^d \Delta_{s_i}^i.$$

Consider the recovery operator (Smolyak operator)

$$T_n := \sum_{s: \|s\|_1 \leq n} \Delta_s.$$

Operator $T_n$ uses $m$ function values with

$$m \ll \sum_{k=1}^n 2^k k^{d-1} \ll 2^n n^{d-1}.$$
First results

The following bound was obtained by S. Smolyak in 1960. Let \( r = (r, \ldots, r) \). In this case write \( W^r_q = W^r_q \). Then

\[
\sup_{f \in W^r_\infty} \| f - T_n \|_\infty \ll 2^{-rn} n^{d-1}, \quad r > 0.
\]

It was extended to the case \( p < \infty \) in VT, 1985:

\[
\sup_{f \in W^r_p} \| f - T_n \|_p \ll 2^{-rn} n^{d-1}, \quad r > 1/p.
\]
The following bound was obtained by S. Smolyak in 1960. Let \( r = (r, \ldots, r) \). In this case write \( W^r_q = W^r_q \). Then

\[
\sup_{f \in W^r_\infty} \| f - T_n \| \infty \ll 2^{-rn} n^{d-1}, \quad r > 0.
\]

It was extended to the case \( p < \infty \) in VT, 1985:

\[
\sup_{f \in W^r_p} \| f - T_n \| p \ll 2^{-rn} n^{d-1}, \quad r > 1/p.
\]

**Open problem.** Find the right order of the optimal sampling recovery \( \varrho_m(W^r_p, L_p) \) in case \( 1 \leq p \leq \infty \) and \( r > 1/p \).
Further results

We have (VT, 1993)

$\varrho_m(\mathcal{W}^r_2)_\infty \approx m^{-r+1/2}(\log m)^{r(d-1)}, \quad r > 1/2.$

The order of optimal recovery is provided by the Smolyak operator $T_n$. 

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Further results

We have (VT, 1993)

$$\varrho_m(W^r_2)_\infty \asymp m^{-r+1/2}(\log m)^{r(d-1)}, \quad r > 1/2.$$  

The order of optimal recovery is provided by the Smolyak operator $T_n$. Also we know (VT, 1993)

$$\sup_{f \in W^r_q} \|f - T_n(f)\|_\infty \asymp 2^{-r(1/q)n} n^{(d-1)(1-1/q)}.$$
For \( \mathbf{s} \in \mathbb{Z}_+^d \) define

\[
\rho(\mathbf{s}) := \{ \mathbf{k} \in \mathbb{Z}^d : [2^{s_j-1}] \leq |k_j| < 2^{s_j}, \quad j = 1, \ldots, d \}
\]

where \([x]\) denotes the integer part of \(x\) and

\[
\delta_{\mathbf{s}}(f)(\mathbf{x}) := \sum_{\mathbf{k} \in \rho(\mathbf{s})} \hat{f}(\mathbf{k}) e^{i(\mathbf{k}, \mathbf{x})}.
\]
For $s \in \mathbb{Z}_+^d$ define

$$\rho(s) := \{k \in \mathbb{Z}^d : [2^{s_j-1}] \leq |k_j| < 2^{s_j}, \quad j = 1, \ldots, d\}$$

where $[x]$ denotes the integer part of $x$ and

$$\delta_s(f)(x) := \sum_{k \in \rho(s)} \hat{f}(k)e^{i(k,x)}.$$

Let an array $\varepsilon = \{\varepsilon_s\}$ be given, where $\varepsilon_s \geq 0$, $s = (s_1, \ldots, s_d)$, and $s_j$ are nonnegative integers, $j = 1, \ldots, d$. 
We denote by $G(\varepsilon, q)$ and $F(\varepsilon, q)$ the following sets of functions ($1 \leq q \leq \infty$):

$$G(\varepsilon, q) := \{ f \in L_q : \|\delta_s(f)\|_q \leq \varepsilon_s \text{ for all } s \},$$

$$F(\varepsilon, q) := \{ f \in L_q : \|\delta_s(f)\|_q \geq \varepsilon_s \text{ for all } s \}.$$
Theorem (VT, 1986)

The following relations hold:

\[
\sup_{f \in G(\varepsilon, q)} \| f \|_p \asymp \left( \sum_s \varepsilon_s^p 2^{s_1(p/q-1)} \right)^{1/p}, \quad 1 \leq q < p < \infty; \\
\inf_{f \in F(\varepsilon, q)} \| f \|_p \asymp \left( \sum_s \varepsilon_s^p 2^{s_1(p/q-1)} \right)^{1/p}, \quad 1 < p < q \leq \infty,
\]

with constants independent of \( \varepsilon \).
Remark (Dinh Zung, 1991; VT, 1993)

In the proof of first relation of Theorem (VT, 1986) we used only the property $\delta_s(f) \in \mathcal{T}(2^s, d)$. That is, if

$$f = \sum_s t_s, \quad t_s \in \mathcal{T}(2^s, d),$$

then for $1 \leq q < p < \infty$,

$$\|f\|_p \leq C(q, p, d) \left( \sum_s \|t_s\|_q^2 \|s\|_1^{(p/q-1)} \right)^{1/p}.$$
For \( s \in \mathbb{N}_0 \) define the univariate operators

\[
A_s := V_{2^s} - V_{2^{s-1}}, \quad V_{1/2} = 0
\]

and for \( s = (s_1, \ldots, s_d) \in \mathbb{N}_0^d \)

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A_s := \prod_{i=1}^d A_{s_i}^i.
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For $s \in \mathbb{N}_0$ define the univariate operators

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and for $s = (s_1, \ldots, s_d) \in \mathbb{N}_0^d$

$$A_s := \prod_{i=1}^{d} A_{s_i}^i.$$  

$$H^r_p := \{ f : \| A_s(f) \|_p \leq 2^{-r}\| s \|_1 \}.$$
Recovery of $H$ classes

Theorem (VT, 1985)

Let $1 \leq p \leq \infty$ and $r > 1/p$. Then we have for $f \in H^r_p$

$$
\|s(f)\|_p \ll 2^{-r} \|s\|_1 \quad \text{and} \quad \|f - T_n(f)\|_p \ll 2^{-rn} n^{d-1}.
$$
Recovery of $H$ classes

Theorem (VT, 1985)

Let $1 \leq p \leq \infty$ and $r > 1/p$. Then we have for $f \in H^r_p$

$$\|\Delta_s(f)\|_p \ll 2^{-r}\|s\|_1 \quad \text{and} \quad \|f - T_n(f)\|_p \ll 2^{-rn}n^{d-1}.$$  

The above Theorem (VT, 1985), Theorem (VT,1986) and remark to it imply:

Theorem (Dinh Zung, 1991; VT, 1993)

For any $f \in H^r_q$, $1 \leq q < p < \infty$, $r > 1/q$

$$\|f - T_n(f)\|_p \ll 2^{-n(r-\beta)}n^{(d-1)/p}, \quad \beta := 1/q - 1/p.$$
It easily follows from the definition of $\varrho_m(F)_p$ that $\varrho_m(F)_p \geq d_m(F, L_p)$, where $d_m(F, L_p)$ is the Kolmogorov width. The upper bound from Theorem (VT, 1985) and the lower bound for the Kolmogorov width from VT, 1998: for $d = 2$

$$d_m(H^r_\infty, L_\infty) \asymp m^{-r}(\log m)^{r+1}$$

imply for $d = 2$

$$\varrho_m(H^r_\infty)_{\infty} \asymp m^{-r}(\log m)^{r+1}.$$
For $N \in \mathbb{N}$ define the hyperbolic cross

$$\Gamma(N) := \{k \in \mathbb{Z}^d : \prod_{j=1}^{d} \max(|k_j|, 1) \leq N\}$$

and the corresponding Dirichlet kernel

$$\mathcal{D}_N(x) := \sum_{k \in \Gamma(N)} e^{i(k,x)}.$$

Consider the hyperbolic cross partial sums

$$S_N(f, x) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(y) \mathcal{D}_N(x - y) dy.$$
It is known that

\[
\sup_{f \in W^r_2} \| f - S_N(f) \|_2 \asymp d_{|\Gamma(N)|}(W^r_2, L^2) \asymp N^{-r}.
\]

For a point set \( \xi(m) = \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d \) consider a discretization of the convolution operator \( S_N \)

\[
S_N(f, \xi(m), x) := \frac{1}{m} \sum_{\nu=1}^m f(\xi^\nu)D_N(x - \xi^\nu).
\]

How many points do we need to guarantee

\[
\sup_{f \in W^r_2} \| f - S_N(f, \xi(m)) \|_2 \asymp d_{|\Gamma(N)|}(W^r_2, L^2) \asymp N^{-r}?
\] (2)
It is known that

\[
\sup_{f \in W_2^r} \| f - S_N(f) \|_2 \asymp d_{|\Gamma(N)|}(W_2^r, L_2) \asymp N^{-r}.
\]

For a point set \( \xi(m) = \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d \) consider a discretization of the convolution operator \( S_N \)

\[
S_N(f, \xi(m), x) := \frac{1}{m} \sum_{\nu=1}^m f(\xi^\nu) D_N(x - \xi^\nu).
\]

How many points do we need to guarantee

\[
\sup_{f \in W_2^r} \| f - S_N(f, \xi(m)) \|_2 \asymp d_{|\Gamma(N)|}(W_2^r, L_2) \asymp N^{-r}?
\] (2)

It is proved in VT, 1986 that it is sufficient to take \( m \asymp N^2 (\log N)^{d-1} \) for (2) to hold. The proof uses number theoretical constructions.
Thank you!

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