Universal discretization for the lower sets

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Outline of the talk

Part 1. Universal discretization:

Definitions

General theorem on universal discretization

Part 2. On cardinality of the lower sets:

Related studies:

- integer partitions (number theory),
- quantum states (stats mechanics),
- lattice animals, polyominos (combinatorics),
- integer-valued monotone functions (analysis)

Old and new results

Some proofs

Universal discretization

Consider the space $L_2(K)$ endowed with some probability measure on the compact set $K \in \mathbb{R}^d$. Given an n-dimensional subspace $\mathcal{U}_n \subset L_2(K)$, the Marcinkievicz discretization problem consist of finding a discrete point set $\xi = \{\xi_i\}_{i=1}^m$ such that, for any $f \in \mathcal{U}_n$, we have

$$c_1 \|f\|_2^2 \leq \frac{1}{m} \sum_{i=1}^{m} |f(\xi_i)|^2 \leq c_2 \|f\|_2^2 .$$

Universal discretization set

Now, suppose that we are given a collection of n-dimensional subspaces $\mathcal{U}_A := \{\mathcal{U}_n^\alpha\}_{\alpha \in A}$. We call a point set $\xi = \{\xi_i\}_{i=1}^m$ a universal discretization set if, for any $\alpha \in A$ and for any $f \in \mathcal{U}_n^\alpha$, we have

$$c_1 \|f\|_2^2 \leq \frac{1}{m} \sum_{i=1}^{m} |f(\xi_i)|^2 \leq c_2 \|f\|_2^2 .$$

Universal discretization

Theorem A (Dai, Prymak, Temlyakov, Tikhonov, 2018)

Suppose that each $\mathcal{U}_n^\alpha \subset \mathcal{U}_A$ is spanned by an orthonormal system $\{u_i^\alpha\}$ that satisfies the so-called Condition E: there exists a constant B such that

$$\sum_{i=1}^{n} |u_i^\alpha|^2 \le Bn.$$

Then there exists a universal discretization set $\xi = \{\xi_i\}_{i=1}^m$ such that, for any $\alpha \in A$ and for any $f \in \mathcal{U}_n^\alpha$, we have

$$c_1\|f\|_2^2 \le \frac{1}{m}\sum_{i=1}^{m} |f(\xi_i)|^2 \le c_2\|f\|_2^2 ,$$

where

$$m \le cn\ln(nC_A)\,, \qquad C_A := \operatorname{card}\mathcal{U}_A\,.$$

Lower sets

Definition 1 (Lower sets)

A lower set Q in $\mathbb{R}^d$ is a set of non-negative integer points $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}_+^d$ such that

$$\mathbf{k} = (k_1, \ldots, k_d) \in Q \quad \Rightarrow \quad \mathbf{k}' \in Q \quad \text{if} \quad 0 \le k_i' \le k_i \quad \forall i\,.$$

We denote by $\mathcal{L}_d(n)$ the set of all lower sets in $\mathbb{R}^d$ of size n,

$$\mathcal{L}_d(n) := \{Q \in \mathbb{Z}_+^d : Q \text{ is a lower set, and } |Q| = n\}\,,$$

and by $p_d(n)$ cardinality of $\mathcal{L}_d(n)$,

$$p_d(n) := \#\mathcal{L}_d(n)\,.$$

These sets are used in multivariate approximation in $\mathbb{R}^d$ by algebraic and trigonometric polynomials which belong to the n-dimensional subspaces spanned by multinomials whose powers form a lower set (of size n).

Positive lower sets

Sometimes, it is more convenient to consider positive lower sets, i.e., the sets which are obtained from the lower sets with the shift by the unit vector $(1, 1, \dots, 1) = e_1 + e_2 + \dots + e_d$.

Definition 2 (Positive lower sets)

A positive lower set is a set of positive integer points $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$ such that

$$\mathbf{k} = (k_1, \dots, k_d) \in P \quad \Rightarrow \quad \mathbf{k}' \in P \quad \text{if} \quad 1 \le k_i' \le k_i \quad \forall i$$

and $|P| = n$.

In other words, if an integer point $\mathbf{k} \in \mathbb{N}^d$ belongs to a positive lower set P, then the hypercube $H_{\mathbf{k}} = \prod_{i=1}^{d} [0, k_i] \cap \mathbb{N}^d$ also belongs to P. Thus, any positive lower set P is a union of some finite number of hypercubes in $\mathbb{N}^d$.

Universal discretization for lower sets in $\mathbb{T}^d$

Our goal is to find cardinality m of a universal discretization set for trigonometric polynomials in $\mathbb{T}^d$ whose harmonics form a lower set in $\mathbb{R}^d$ of size n.

By Theorem A, this problem is reduced to finding cardinality of $\mathcal{L}_d(n)$, the set of all lower sets of size n.

1) Elementary estimate. Any lower set $Q \in \mathcal{L}_d(n)$ belongs to the hypercube $[0, n]^d$, hence for cardinality of entire collection $\mathcal{L}_d(n)$ we have

$$p_d(n) \le \binom{n^d}{n} \le n^{dn} \quad \Rightarrow \quad m \le Cn \ln p_d(n) \le Cn^2 \ln n \cdot d \,.$$

2) Better estimate. Any lower set $Q \in \mathcal{L}_d(n)$ belongs to the hyperbolic cross

$$\Gamma_d(n) := \{\mathbf{k} \in \mathbb{Z}_+ : \prod_{i=1}^{d}(k_i + 1) \leq n\}.$$

The following estimate for its cardinality is known

$$C_\Gamma := |\Gamma_d(n)| \leq cn^{2+\log_2 d},$$

so for the lower sets of size n we get

$$p_d(n) \leq \binom{C_\Gamma}{n} \leq \left(\frac{C_\Gamma}{n}\right)^n,$$

and respectively for a universal discretization set

$$m \leq cn \ln p_d(n) \leq cn^2 \ln n \ln d.$$

Integer partitions

Question

Can we do better than that?

With a slightly different definition, the positive lower sets in $\mathbb{N}^d$ are well-known in number theory as integer partitions, with a vast literature on this subject (though restricted mainly to the cases $d = 2$ and $d = 3$).

Definition 3 (Integer partition in $\mathbb{R}^d$)

An integer partition P of $n \in \mathbb{N}$ in $\mathbb{R}^d$ is any representation of n in the form

$$n = \sum_{\mathbf{k} \in \mathbb{N}^{d-1}} n_{\mathbf{k}}, \qquad \text{where} \quad n_{\mathbf{k}} \in \mathbb{N}$$

$$\text{and} \quad n_{\mathbf{k}'} \geq n_{\mathbf{k}} \quad \text{if} \quad 1 \leq k'_i \leq k_i \quad \forall i, \quad 1 \leq i \leq d-1\,.$$

Integer partitions and positive lower sets

We may visualize an integer partition of $n \in \mathbb{N}$ in $\mathbb{R}^d$ as a union of stacks of d-dimensional unit cubes, where each stack consists of $n_{\mathbf{k}}$ cubes one over another, with the lowest one having the $(d-1)$-dimensional base centered at $\mathbf{k} - \frac{1}{2}$.

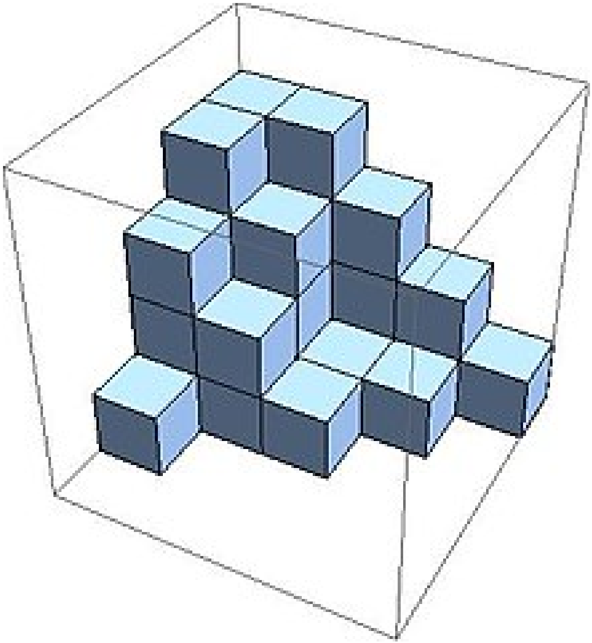
Now, with any d-dimensional partition of n,

$$n = \sum_{\mathbf{k} \in \mathbb{N}_{d-1}} n_{\mathbf{k}} \,,$$

we can identify the set of points $\mathbf{k}' \in \mathbb{N}^d$ by the rule

$$\mathbf{k} = (k_1, \ldots, k_{d-1}) \in \mathbb{N}_{d-1} \quad \to \quad \mathbf{k}' = (k_1, \ldots, k_{d-1}, n_{\mathbf{k}}) \in \mathbb{N}_d \,,$$

and clearly this gives a one-to-one correspondence between integer partitions of n in $\mathbb{R}^d$ and positive lower sets of size n in $\mathbb{R}^d$.



Hardy-Ramanujan formula

Besides number theory, integer partitions also prominently appear in statistical mechanics, and in studying the so-called lattice animals. In fact, all known results about cardinality of $\mathcal{L}_d(n)$ for high dimensions $d > 3$ were obtained in these areas.

Linear partitions ($d = 2$). A linear or two-dimensional partition of a positive integer $n \in \mathbb{N}$ is given by a formula

$$n = n_1 + n_2 + \cdots + n_k \,, \qquad \text{where } n_i \geq n_{i+1}.$$

The asymptotic number of all such partitions $p_2(n)$ as $n \to \infty$ is given by the Hardy-Ramanujan result

$$p_2(n) \sim \frac{1}{4n\sqrt{3}} \exp(c\sqrt{n}), \qquad c = \pi\sqrt{\frac{2}{3}}\,,$$

so that

$$\lim_{n\to\infty} \frac{\ln p_2(n)}{n^{1/2}} = \pi\sqrt{\frac{2}{3}} = 2.565099.$$

Plane partitions

Similarly, a plane or three-dimensional partition of n is given by a formula

$$n = \sum_{i,j} n_{i,j},$$

$$\text{where} \quad n_{i,j} \geq n_{i+1,j} \quad \text{and} \quad n_{i,j} \geq n_{i,j+1} \; .$$

In this case the asymptotic is also known

$$p_3(n) \sim c_1 \left(\frac{n}{2}\right)^{-25/36} \exp\left(3\zeta(3)^{1/3}\left(\frac{n}{2}\right)^{2/3}\right),$$

with the following numerical value

$$\lim_{n\to\infty} \frac{\ln p_3(n)}{n^{2/3}} = 2.00945\,.$$

Solid partitions

Finally, a solid or four-dimensional partition of n is given by a formula

$$n = \sum_{i,j,k} n_{i,j,k}, \quad \text{where} \quad n_{i,j,k} \geq \max\left(n_{i+1,j,k}, n_{i,j+1,k}, n_{i,j,k+1}\right).$$

In that case, the exact asymptotic is not known. A formula suggested by Mahon for the generating function of $p_d(n)$ would provide

$$\alpha_d := \lim_{n\to\infty} \frac{\ln p_d(n)}{n^{1-1/d}} = \frac{d}{d-1}\Big[(d-1)\zeta(d)\Big]^{1/d} =: \beta_d\,,$$

however this formula was disproved for $d > 3$. There was a conjecture that the RHS still provides the right asymptotics, but Monte-Carlo simulations for $d = 4$ produced inconclusive results, namely

$$\beta_4 = 1.78982, \quad \lim_{n\to\infty} \ln p_4(n) n^{-3/4} = 1.822.$$

Integer partitions for $d > 4$

Theorem A (Bhatia-Prasad-Arora, 1997)

For a fixed d and for sufficiently large n,

$$C_1(d) \leq \frac{\ln p_d(n)}{n^{1-1/d}} \leq C_2(d)\,,$$

with some constants C_1, C_2 that depend on d,

However, they did not establish the nature of the constants' dependence on d, and actually did not mention such a dependence at all.

New results

As we have mentioned, with exception of $d = 2$, all the estimates for cardinality of integer partitions are asymptotic - with a fixed dimension d and the number of elements n in partition going to infinity.

We were aiming at getting results with explicit constants and explicit relations between d and n.

The first result provides lower and upper bounds which are uniform in d and n.

Theorem 1

For any $d \geq 2$ and $n \in \mathbb{N}$, we have

$$\frac{1}{(n-1)!} d^{n-1} \leq p_d(n) \leq d^{n-1} .$$

In the next result, we follow the same scheme as suggested in [Bhatta et al], but we put some rigour and care in order to get explicit behaviour of the constants involved.

Theorem 2

For any $d \geq 2$ and $n \in \mathbb{N}$, we have

$$\gamma_d' \leq \frac{\ln p_d(n)}{n^{1-1/d}} \leq \gamma_d$$

where

$$\gamma_d = \gamma_2^{\left(d^{c \ln d}\right)}, \qquad \gamma_d' = \frac{d}{(d!)^{1/d}} \ln 2 \,.$$

We note that

$$\lim_{d \to \infty} \gamma_d' = e \ln 2 = 1.8841 > 1.822 =: \alpha_4,$$

and that means that either the value of α_4 is wrong, or if it is correct, then the numbers $\alpha_d := \lim_{n \to \infty} \frac{\ln p_d(n)}{n^{1-1/d}}$ do not decrease monotonely in d.

Proof of Theorem 1

Theorem 1a

We have

$$p_d(n) \leq d^{n-1}$$

Proof. The proof is by induction on d.

1) For $d = 2$, we may partition n as follows. Firstly we write

$$n = \underbrace{1 + 1 + 1 \cdots + 1 + 1}_{n \text{ times}}$$

and then we form a partition of n in k summands as $n = n_1 + n_2 + \cdots + n_k$ by putting $k - 1$ splits in the sum above

$$n = \underbrace{1 + \cdots + 1}_{n_1} \overset{1}{\oplus} \underbrace{1 + \cdots + 1}_{n_2} \overset{2}{\oplus} \cdots \underbrace{1 + \cdots + 1}_{n_{k-1}} \overset{k-1}{\oplus} \underbrace{1 + \cdots + 1}_{n_k}$$

For each k we have $\binom{n-1}{k-1}$ places where to put the split in, so the total number of such partitions is clearly $\sum_{k=1}^{n-1} \binom{n-1}{k-1} = 2^{n-1}$. Therefore, since integer partitions are the ordered one, we have

$$p_2(n) \leq 2^{n-1} .$$

Proof of Theorem 1

2) Assuming induction hypothesis to be true for d, we make the cuts of each $(d+1)$-dimensional partition along the hyperplanes perpendicular to the x_1-axis, say. Let there be $k-1$ such cuts, with each d-dimensional part contaning n_i points. Then the total number of $(d+1)$-dimensional partitions can be estimated as follows,

$$
\begin{aligned}
p_{d+1}(n) &\leq \sum_{k=1}^{n-1} \binom{n-1}{k-1} \prod_{i=1}^{k} p_d(n_i) \\
&\leq \sum_{k=1}^{n-1} \binom{n-1}{k-1} \prod_{i=1}^{k} d^{n_i-1} \\
&= \sum_{k=1}^{n-1} \binom{n-1}{k-1} d^{n-k} \\
&= \sum_{k=1}^{n-1} \binom{n-1}{k-1} d^{(n-1)-(k-1)} \cdot 1^{k-1} \\
&= (d+1)^{n-1}.
\end{aligned}
$$

Lemma

Let $w_d(m)$ be the number of ways of distributing m balls between d ordered boxes. Then

$$w_d(m) = \binom{m + (d-1)}{d-1}$$

Proof. One can encode every single distribution with a sequence of $d-1$ zeros and m ones.

Theorem 1b

We have

$$p_d(n) \geq \binom{(d-1) + (n-1)}{n-1} \geq \frac{1}{(n-1)!}\, d^{n-1}\,.$$

Proof. We obtain the lower bound as cardinality of the subset of lower sets where all the points lie on coordinate axes. This means that we put the first point at the origin, and then putting remaining $n-1$ points along d axes. This gives the required bound using the lemma above.

Proof of Theorem 2b: the lower bound

Theorem 2b

For any $d \geq 2$ and $n \in \mathbb{N}$, we have

$$\frac{\ln p_d(n)}{n^{1-1/d}} \geq \gamma'_d = \lambda_d \ln 2\,, \qquad \lambda_d := \frac{d}{(d!)^{1/d}} \to e\,.$$

Proof. Given $m \in \mathbb{N}$ consider the sets

$$A_m := \{\mathbf{k} \in \mathbb{Z}_+^d : k_1 + k_2 + \cdots + k_d = m\}\,,$$

and

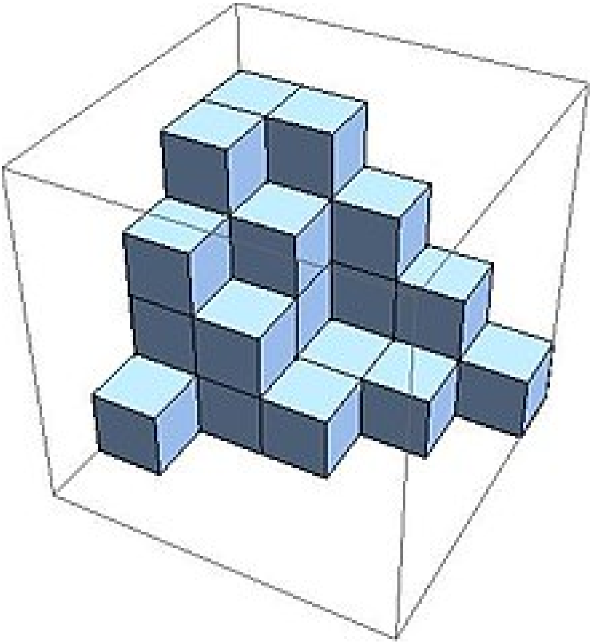
$$B_m := \{\mathbf{k} \in \mathbb{Z}_+^d : k_1 + k_2 + \cdots + k_d \leq m\}\,.$$

Clearly, B_m is a lower set, and A_m is the set of its corners (or vertices). Let a_m and b_m be the cardinality of A_m and B_m, respectively,

$$a_m := \#A_m, \qquad b_m := \#B_m\,.$$

Then a_m is the number of ways of putting m balls into d boxes, hence

$$a_m = \binom{m + (d-1)}{(d-1)}, \qquad b_m = \sum_{s=0}^{m} a_s = \binom{m+d}{d}$$



Proof of Theorem 2b: the lower bound (contd)

It follows then that

$$a_m > \lambda_d b_{m+1}^{1-1/d}$$

Next, given $n \in \mathbb{N}$, we choose m such that

$$b_m \leq n < b_{m+1}$$

and with such an n we take the lower set B_m of cardinality b_m and put all the remaining points along, say, the x_1-axis. If we remove any number of corners (vertices) of B_m, the remaining set remains a lower set, so collection of all such sets (where any corner is either in or out) has cardinality 2^{a_m}. This gives

$$\ln p_d(n) > a_m \ln 2 > \lambda_d b_{m+1}^{1-1/d} \ln 2 > \lambda_d n^{1-1/d} \ln 2 \,.$$

Proof of Theorem 2a: the upper bound

Theorem 2a

For any $d \geq 2$ and $n \in \mathbb{N}$, we have

$$\frac{\ln p_d(n)}{n^{1-1/d}} \leq \gamma_d, \qquad \gamma_d = \gamma_2^{\left(d^{c \ln d}\right)}.$$

Proof. The proof is by induction on d, the hypothesis being true for $d = 2$.

For a lower set A of size n, let m_1 be the largest integer such that A contain the point $\mathbf{m} = (m_1, \ldots, m_1)$. Then

$$m_1 \leq \lfloor n^{1/d} \rfloor.$$

Then A is contained in the union of $m_2 = dm_1$ hyperplanes

$$\{x_1 = 1\}, \ldots, \{x_d = 1\}, \quad \cdots \quad \{x_1 = m_1\}, \ldots, \{x_d = m_1\}.$$

Next we start to cut A into $(d-1)$-dimensional slices choosing each time the hyperplane which contains the largest number of points from A.

Proof of Theorem 2a: the upper bound (contd)

The number of ways the m_2 hyperplanes are chosen. This is equal to

$$\frac{(dm_1)!}{(m_1!)^d} \leq \frac{(dm_1)^{dm_1}}{(m_1)^{dm_1}} \frac{(dm_1)^{1/2}}{m_1^{d/2}} \leq d^{dm_1}$$

We obtain then

$$p_d(n) \leq \sum \prod_{i=1}^{m_2} p_{d-1}(n_i) d^{m_2}$$

The sum extends over all values of n_i which form a partition of n, i.e., $n_i \geq n_{i+1}$ and $\sum n_i = n$, and by induction

$$\sum 1 \leq \gamma_2^{n^{1/2}}$$

Proof of Theorem 2a: the upper bound (contd)

For the product, we obtain

$$\prod_{i=1}^{m_2} p_{d-1}(n_i) \le \prod_{i=1}^{m_2} (\gamma_{d-1})^{n_i^{1-\frac{1}{d-1}}} = \gamma_{d-1}^{s},$$

where (by Hölder inquality)

$$s = \sum_{i=1}^{m_2} n_i^{1-1/(d-1)} \le n^{1-1/d} d^{1/(d-1)}\,.$$

So, we obtain

$$p_d(n) \le \gamma_2^{n^{1/2}} d^{dn^{1/d}} (\gamma_{d-1})^{d^{1/(d-1)} \cdot n^{1-1/d}} \le \gamma_d^{n^{1-1/d}},$$

$$\gamma_d = \gamma_2^{r}, \qquad r = \prod_{k=2}^{d} (k+1)^{1/k-1} \le d^{c \ln d}\,.$$

Summary

Theorem

For any n, there is a universal discretization set $\xi = (\xi_i)_{i=1}^m$ such that for any trigonometric polynomial on T^d, with harmonics forming a lower set of size n, we have

$$c_1 \|f\|_2^2 \le \frac{1}{m} \sum_{i=1}^{m} |f(\xi_i)|^2 \le c_2 \|f\|_2^2 ,$$

where cardinality m of ξ satisfies the upper bounds below.

$$\begin{array}{lllll}
1) & p_d(n) \le n^{dn} & \Rightarrow & m \le cn^2 \ln n \cdot d, \\
2) & p_d(n) \le n^{n(1+\log d)} & \Rightarrow & m \le cn^2 \ln n \cdot \ln d, \\
3) & p_d(n) \le d^n & \Rightarrow & m \le cn^2 \ln d, \\
4) & p_d(n) \le (\gamma_2)^{rn^{1-1/d}} & \Rightarrow & m \le cn^{2-1/d} d^{c_1 \ln d}.
\end{array}$$