Strict density inequalities for sampling and interpolation sets in weighted spaces of holomorphic functions

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Weighted Banach spaces of entire functions in $\mathbb{C}^n$

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- The space A^2_ϕ is reproducing kernel Hilbert space. We denote the kernel K_ϕ. No explicit formula in general.

Sampling and interpolation

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- A sampling sequence is equivalent to knowing wether translatations and modulations of the Gaussian are a frame in $L^2(\mathbb{R}^n)$.
- Intuitively, interpolating sets are sparse, sampling sets are dense.

Densities

- We try to understand sampling and interpolating sets in terms of densities. To this end we define upper density

$$D_K^+(\Lambda) = \limsup_{R\to\infty} \sup_{x\in\mathbb{C}^n} \frac{\sharp(\Lambda \cap B(x,r))}{\int_{B(x,r)} K_\phi(w,w)e^{-2\phi(w)}dm(w)}$$

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- We need also another pair of densities:

$$D_{\partial\bar\partial\phi}^+(\Lambda) = \limsup_{R\to\infty} \sup_{x\in\mathbb{C}^n} \frac{\sharp(\Lambda\cap B(x,r))}{\int_{B(x,r)}(i\partial\bar\partial\phi(w))^n},$$

where $(i\partial\bar\partial\phi)^n = \det(\partial_j\bar\partial_k\phi(z))_{1\le j,k\le n}dm(z)$. $D_{\partial\bar\partial\phi}^-(\Lambda)$ defined analogously.

Previous results

- We will from now on assume that $A \cdot Id \leq (\partial_j \bar{\partial}_k \phi)_{1 \leq j,k \leq n} \leq B \cdot Id$ in the sense of pos-def matrices. General necessary density condition à la Landau: $D_K^-(\Lambda) \geq 1$ for sampling sets and $D_K^+(\Lambda) \leq 1$ for interpolating sets.

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- In 1D, there is a complete characterization: $D^-_{\partial\bar{\partial}\phi}(\Lambda) > 2/\pi$ iff Λ is sampling and $D^+_{\partial\bar{\partial}\phi}(\Lambda) < 2/\pi$ iff Λ is interpolating. Notice that here we use a different density!

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- With a little work, combining these results shows that in 1D, the two densities are equal (modulo constant factor). Also true when $n \geq 2$ when ϕ is 2-homogeneous (Lindholm).
- Complete characerization of sampling and interpolation not possible in terms of Beurling density when $n \geq 2$.

Balian-Low type strict density inequality

Theorem (Gröchenig, Haimi, Ortega-Cerda, Romero)

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- Cf. Balian-Low in time-frequency analysis.
- In $1 - D$, this was proved by using certain arguments involving specifically $1 - D$ complex analysis.
- We also show that this is sharp by constructing a critical set Λ such that Λ is interpolating for $A^2_{(1+\epsilon)\phi}$ and Λ is sampling for $A^2_{(1-\epsilon)\phi}$.

Translated weights

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- For $\lambda \in \mathbb{C}^n$, define a certain "translated" weight ϕ_λ, as a certain solution of the equation

$$\partial\bar{\partial}\phi_\lambda = \partial\bar{\partial}\phi(\cdot - \lambda).$$

  Because the form $\partial\bar{\partial}\phi$ is bounded from above uniformly, we can employ an $\bar{\partial}$ estimate of Berndtsson-Andersson and Poincare lemma to show that

$$\phi_\lambda(z) \le C|z|^2 \log(|z|^2 + 1)$$

  for a constant C independent of λ. The exact growth rate in z is irrelevant but uniformity in λ is the key.

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- Let $q(z,\lambda)$ be a holomorphic function z satisfying

$$\mathrm{Re} q(z,\lambda) = \phi_\lambda - \phi(\cdot - \lambda)$$

(right hand side is harmonic because of the def of ϕ_λ).

- Define

$$T_\lambda f(z) = f(z-\lambda)e^{q(z,\lambda)}.$$

A small computation shows that

$$T_\lambda : A^p_\phi \to A^p_{\phi_\lambda}$$

is an isometric isomorphism. These are our generalized translation operators.

Sampling and interpolation sets same for all p

- An important part of the proof is to show that sampling and interpolation sets are the same for all p, i.e. if Λ is sampling (or interpolation) for A^p_ϕ for some $p \in [1, \infty]$, then it is sampling (or interpolation) for all $p \in [1, \infty]$.

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- This is done by applying theory of localized frames and an extended version of Sjöstrand's Wiener type lemma.
- This part does not rely on we dealing with analytic functions.

Outline of the proof

- Proof uses Beurling's basic method of ”weak limits” and adds Sjöstrand's Wiener type lemma. The setting is not translation invariant: we replace these by the translation type operators.

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- Let's discuss lower bound in sampling. Suppose the result is not true. Suppose there is a sampling set Λ s.t. $D_K^-(\Lambda) = 1$. Then by the non-strict condition, $\Lambda_j := (1 + 1/j)\Lambda$ is not sampling for any j. Naive attempt: take functions $\|f_j\| = 1$ whose energy on the set Λ_j is small and find a limiting function f which vanishes on Λ, therefore violating the assumption that Λ is sampling.

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- Doesn't work! Because how would we know that f is not identically zero?

More on the proof

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- Suppose the space is translation invariant (generally not true). Then we would translate f_j so that the value at the origin is $\geq 1/2$ and then find a limiting function f. By construction $|f(0)|e^{-2\phi(0)} > 1/2$ so it's not identically zero and the convergence is uniform on compacts.

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- This function f vanishes on some set Γ. It can be shown that Γ arises as a weak limit of the translates of the original set Λ and that Γ is sampling because Λ is. We have a contradiction!

Construction of near-critical sets, part I

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$$\Delta(\lambda_1^{(n)}, \dots, \lambda_n^{(n)}) := \det[p_j^{(n)}(\lambda_k^{(n)})]_{1 \leq j,k \leq n} e^{-\phi(\lambda_1^{(n)}) - \dots - \phi(\lambda_n^{(n)})}.$$

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- We will need the Lagrange interpolants $l_j^{(n)}, j = 1, \ldots, n$ related to Λ_n.

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- The sets $\Lambda^{(n)}$ possess a subsequence that has as its weak limit some separated set Λ. By passing to a further subsequence, we find interpolating functions l_λ as a limit of the Lagrange interpolants, solving the interpolation problem $l_\lambda(\lambda) = 1$, $l_\lambda(\mu) = 0$ for $\mu \in \Lambda, \mu \neq \lambda$.

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- The limit functions satisfy $\|l_\lambda\|_\infty \leq 1$. Therefore given some sequence $(c_\lambda) \in l^1(\Lambda)$, we define $l := \sum_\lambda c_\lambda l_\lambda$. This function satisfies $l(\lambda) = c_\lambda$, with

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- The function is l is not in the desired space yet. We improve the localization as follows

$$\tilde{l}_\lambda := l_\lambda(z) \frac{K_{\epsilon\phi}(z, \lambda) e^{-\epsilon\phi(z) - \epsilon\phi(\lambda)}}{K_{\epsilon\phi}(\lambda, \lambda) e^{-2\epsilon\phi(\lambda)}}$$

for some small ϵ.

Construction of near critical sets, part III

The modified function $\sum c_\lambda \tilde{l}_\lambda$ solves the same interpolation problem as before and belongs to $A^1_{(1+\epsilon)\phi} \cdot e^{-(1+\epsilon)\phi}$.

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- For example: in Paley Wiener space, strict density theorems are not true and repeating the same argument as we did doesn't work because of weaker off-diagonal decay.
- The construction of near critical functions is also quite general.