# Sampling discretization and moments of random vectors

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# Sampling discretization problem

- $C > c > 0$ — fixed
- $L \subset L^p := L^p(\Omega, \mu) \cap C(\Omega)$ — $N$-dimensional

**Main question:**

For what $m \in \mathbb{N}$ there are $X_1, \dots, X_m \in \Omega$ such that

$$c\|f\|_p^p \le \frac{1}{m}\sum_{j=1}^{m} |f(X_j)|^p \le C\|f\|_p^p \quad \forall f \in L?$$

Here $\|f\|_p := \Big(\int_\Omega |f|^p \, d\mu\Big)^{1/p}$.

# Comments

- $m \geq N \Rightarrow$ we are interested in the conditions on $L$ under which $m$ is close to $N$

- **Sampling discretization with weights:** For what $m \in \mathbb{N}$ there are $X_1, \dots, X_m \in \Omega$ and numbers $\lambda_1, \dots, \lambda_m$ such that

$$c\|f\|_p^p \leq \sum_{j=1}^{m} \lambda_j |f(X_j)|^p \leq C\|f\|_p^p \quad \forall f \in L?$$

- Special case: $C = 1 + \varepsilon$, $c = 1 - \varepsilon$, $\varepsilon > 0$

- Further: $p > 1$

# Probabilistic approach

- We choose the points $X_1, \dots, X_m$ randomly, i.e. $X_1, \dots, X_m$ are i.i.d. r.v. with the distribution $\mu$.

- For $B \subset L$ consider the random variable

$$V_p(B) := \sup_{f \in B} \left| \frac{1}{m} \sum_{j=1}^{m} |f(X_j)|^p - \|f\|_p^p \right|.$$

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• If $B = B_p(L) := \{f \in L \colon \|f\|_p \le 1\}$ and $P(V_p(B_p(L)) \le \varepsilon) > 0$ then

$$(1-\varepsilon)\|f\|_p^p \le \frac{1}{m} \sum_{j=1}^{m} |f(X_j)|^p \le (1+\varepsilon)\|f\|_p^p \quad \forall f \in L.$$

• $P\big(V_p(B) \le 2\mathbb{E}\big[V_p(B)\big]\big) \ge 2^{-1}$
$\Rightarrow$ sufficient to bound $\mathbb{E}\big[V_p(B)\big]$

## Connection with approximation of moments of random vectors

- Let $\mathbf{u} = (u_1, \dots, u_N)$ be a random vector in $\mathbb{R}^N$;
- $\langle \cdot, \cdot \rangle$ — inner product in $\mathbb{R}^N$;
- $K \subset \mathbb{R}^N$;
- $U_p(K) := \sup_{y \in K} \left| \frac{1}{m} \sum_{j=1}^{m} |\langle y, \mathbf{u}^j \rangle|^p - \mathbb{E}|\langle y, \mathbf{u} \rangle|^p \right|.$

**The main question:** How many independent copies $\mathbf{u}^1, \dots, \mathbf{u}^m$ of $\mathbf{u}$ are needed to guarantee $U_p(K) \le \varepsilon$ with high probability?

- Extensively studied: M. Rudelson, R. Vershynin, K. Tikhomirov, O. Guédo, R. Adamczak, A. Litvak, A. Pajor, N. Tomczak-Jaegermann,...

- If $K \subset \mathbb{R}^N$
$\Rightarrow$ take $B := \{f_y(\cdot) = \langle y, \cdot\rangle \colon y \in K\}$, $\mu = Law(\mathbf{u})$,
$\Rightarrow \mathbb{E}\big[V_p(B)\big] = \mathbb{E}\big[U_p(K)\big]$.

- If $\langle\cdot,\cdot\rangle$ — any inner product on $L \subset L^p$, $B \subset L$
$\Rightarrow$ take an orthonormal basis $u_1, \dots, u_N$ in $L$,
take $\mathbf{u}^j := (u_1(X_j), \dots, u_N(X_j))$,
take $K := \Big\{y = (y_1, \dots, y_N) \in \mathbb{R}^N \colon \sum_{k=1}^{N} y_k u_k \in B\Big\}$
$\Rightarrow \mathbb{E}\big[U_p(K)\big] = \mathbb{E}\big[V_p(B)\big]$.

## Some known results: $p = 2$

**Theorem (M. Rudelson, 99).**
Let $K_2 := \{y \in \mathbb{R}^N \colon |y| \le 1\}$ and assume that $|y|^2 = \mathbb{E}|\langle \mathbf{u}, y\rangle|^2$. Then

$$\mathbb{E}\big[U_2(K_2)\big] \le C(A + \sqrt{A}),$$

where $A = \frac{\log N}{m}\mathbb{E}\big[\max_{1\le j\le m} |\mathbf{u}^j|^2\big]$.

**Equivalently:**

$$\mathbb{E}\big[V_2(B_2(L))\big] \le C(A + \sqrt{A}),$$

where $B_2(L) := \{f \in L \colon \|f\|_2 \le 1\}$ and
$A = \frac{\log N}{m}\mathbb{E}\big[\sup_{f\in B_2(L)} \max_{1\le j\le m} |f(X_j)|^2\big]$

# Nikolskii-type inequality assumption

**Definition.** Subspace $L$ satisfies $(\infty, q)$ Nikolskii-type inequality assumption (with constant $M > 0$) if

$$\sup_{x\in\Omega} |f(x)| = \|f\|_\infty \le MN^{1/q}\|f\|_q \quad \forall f \in L.$$

**Corollary.** If $L$ satisfies $(\infty, 2)$ Nikolskii-type inequality assumption, then there are $m = C(\varepsilon, M)\, N \log N$ points $X_1, \dots, X_m$:

$$(1-\varepsilon)\|f\|_2^2 \le \frac{1}{m}\sum_{j=1}^{m} |f(X_j)|^2 \le (1+\varepsilon)\|f\|_2^2 \quad \forall f \in L.$$

## Without randomness: $p = 2$

**Theorem (V. Temlyakov, I. Limonova, 20).**
$\exists\, C_1, C_2, C_3 > 0$: $\forall$ $N$-dimensional subspace $L \subset L^2$, satisfying $(\infty, 2)$ Nikolskii-type inequality assumption, $\exists\, m \le C_1 M^2 N$ points $X_1, \dots, X_m$:

$$C_2 \|f\|_2^2 \le \frac{1}{m} \sum_{j=1}^{m} |f(X_j)|^2 \le C_3 M^2 \|f\|_2^2 \quad \forall f \in L.$$

## Known bounds for $\theta$-convex sets

**Definition.** $B \subset L$ is $\theta$-convex with constant $\zeta > 0$ if

$$\left\|\frac{f+g}{2}\right\|_B \leq 1 - \zeta\|f-g\|_B^\theta \quad \forall f, g \in B.$$

**Theorem (O. Guédon, M. Rudelson, 07).**
If $B \subset L$ is $\theta$-convex and $B \subset D$ — Euclidean ball, then for $p \in [\theta, \infty)$ one has

$$\mathbb{E}\big[V_p(B)\big] \leq C\Big(A + A^{1/2}(\sup_{f\in B}\mathbb{E}|f(X_1)|^p)^{1/2}\Big),$$

$$A = \frac{[\log m]^{2-\frac{2}{\theta}}}{m}\mathbb{E}\Big(\sup_{f\in D}\max_{1\leq j\leq m}|f(X_j)|^2 \sup_{h\in B}\max_{1\leq j\leq m}|h(X_j)|^{p-2}\Big).$$

- $B_p(L)$ is $\max\{p,2\}$-convex;
- $p \geq 2$: $B_p(L) \subset B_2(L) \Rightarrow B = B_p(L)$, $D = B_2(L)$;
- $p \geq 2$, $L$ satisfies $(\infty, 2)$ Nikolskii-type inequality $\Rightarrow A \leq M^p N^{p/2} \frac{[\log m]^{2-\frac{2}{p}}}{m} \Rightarrow$ discretization of $L^p$-norm with $m = CN^{p/2}[\log N]^{2-\frac{2}{p}}$ points $(C = C(M, \varepsilon, p))$;
- $p \geq 2$, $L$ satisfies $(\infty, p)$ Nikolskii-type inequality $\Rightarrow L$ satisfies $(\infty, 2)$ Nikolskii-type inequality with constant $M^{p/2} \Rightarrow A \leq M^{2p-2} N^{2-\frac{2}{p}} \frac{[\log m]^{2-\frac{2}{p}}}{m} \Rightarrow$ discretization of $L^p$-norm with $m = CN^{2-\frac{2}{p}}[\log N]^{2-\frac{2}{p}}$ points $(C = C(M, \varepsilon, p))$.

# The first main result: analog of Guédon–Rudelson

**Theorem (E.K., 21).** If $B \subset L$ is $\theta$-convex, then for $p \in [\theta, \infty)$ one has

$$\mathbb{E}\big[V_p(B)\big] \le C\Big(A + A^{\frac{1}{\theta}}(\sup_{f\in B}\mathbb{E}|f(X_1)|^p)^{1-\frac{1}{\theta}}\Big),$$

where $A = \frac{[\log m]^\theta}{m}\mathbb{E}\Big(\sup_{f\in B}\max_{1\le j\le m}|f(X_j)|^p\Big)$.

**Corollary.**
$p \ge 2$, $L$ satisfies $(\infty, p)$ Nikolskii-type inequality $\Rightarrow$ take $B = B_p(L)$ $\Rightarrow$ discretization of $L^p$-norm with $m = CN[\log N]^p$ points ($C = C(M, \varepsilon, p)$).

## Discretization under the entropy numbers decay assumption

**Definition.** $(F,\varrho)$ — metric space,

$$e_k(F,\varrho):=\inf\Big\{\varepsilon\colon \exists f_1,\dots,f_{n_k}\in F\colon F\subset \bigcup_{j=1}^{n_k} B_\varepsilon(f_j)\Big\},$$

where $n_k=2^{2^k}$, $n_0=1$, $B_\varepsilon(f):=\{g\colon \varrho(f,g)<\varepsilon\}$.

**Theorem (F. Dai, A. Prymak, A. Shadrin, V. Temlyakov, S. Tikhonov, 20).**

Let $p\in[1,\infty)$ and assume that

$$e_k(B_p(L),\|\cdot\|_\infty)\le MN^{1/p}2^{-k/p}\quad 0\le k\le \log N.$$

Then $\exists m\le C(p)M^pN[\log(2MN)]^2$ points such that

$$\tfrac{1}{2}\|f\|_p^p\le \tfrac{1}{m}\sum_{j=1}^{m}|f(X_j)|^p\le \tfrac{3}{2}\|f\|_p^p\quad \forall f\in L.$$

**Definition.** For any fixed set $X = \{X_1, \dots, X_m\}$ let $\|f\|_{\infty,X} := \max_{1\le j\le m} |f(X_j)|$.

**Theorem (E.K., 21).**
Let $\alpha \in (1,\infty)$, $p \in [\alpha,\infty)$, $\theta \ge 2$, $B$ — $\theta$-convex.
Assume that $\forall X = \{X_1, \dots, X_m\}$ $\exists W_B(X) > 0$:

$$e_k(B, \|\cdot\|_{\infty,X}) \le W_B(X) 2^{-k/\alpha}.$$

Then

$$\mathbb{E}\big[V_p(B)\big] \le C\Big(A + A^{\frac{1}{\max\{\alpha,2\}}}\big(\sup_{f\in B}\mathbb{E}|f(X_1)|^p\big)^{1-\frac{1}{\max\{\alpha,2\}}}\Big),$$

where

$$A = \frac{[\log m]^{\max\{\alpha,2\}(1-\frac{1}{\theta})}}{m}\mathbb{E}\Big([W_B(X)]^\alpha \sup_{f\in B}\max_{1\le j\le m}|f(X_j)|^{p-\alpha}\Big).$$

# Corollary for $B_p(L)$

- $B_p(L)$ — $\max\{p,2\}$ — convex;
- take $\alpha = p$, $\theta = \max\{p,2\}$, and assume that

$$e_k(B_p(L), \|\cdot\|_{\infty,x}) \le C(m)N^{1/p}2^{-k/p}.$$

Then $A \le \frac{[\log m]^{\max\{p,2\}-1}}{m}[C(m)]^p N$

- If $C(m) \le M[\log m]^r \Rightarrow$ discretization of $L^p$-norm with $m = CN[\log N]^{\max\{p,2\}-1+pr}$ points ($C = C(M,\varepsilon,p)$).

# Bounds for the entropy numbers

- (M. Talagrand) $B - \theta$-convex $\Rightarrow$

$$e_k(B, \|\cdot\|_{\infty,X}) \le C\Big[\max_{1\le j\le m}\sup_{f\in B}|f(X_j)|\Big][\log m]^{1/\theta}2^{-k/\theta}.$$

- $\Rightarrow$ the analog of Guédon–Rudelson bound and discretization result for $p > 2$ ($m = CN[\log N]^p$ under the $(\infty, p)$ Nikolskii-type ineq. assump.)

## The case $p \in (1,2)$

**Theorem.** $p \in (1,2)$, $L$ satisfies $(\infty, 2)$ Nikolskii-type ineq. assump. with $M \geq 2$, then

$$e_k(B_p(L), \|\cdot\|_{\infty,X}) \leq C[\log m]^{\frac{1}{2}}[\log M^2 N]^{\frac{1}{p}-\frac{1}{2}} M^{\frac{2}{p}} N^{\frac{1}{p}} 2^{-k/p}.$$

**Theorem (E.K., 20).** $p \in (1,2)$, $L$ satisfies $(\infty, 2)$ Nikolskii-type ineq. assump., then

$$\mathbb{E}\left[V_p(B_p(L))\right] \leq C\left(A + \sqrt{A}\right),$$

where $A = \frac{[\log m]^{1+\frac{p}{2}}[\log 4M^2N]^{1-\frac{p}{2}}}{m} M^2 N$.

**Corollary.** $p \in (1,2)$, $L$ satisfies $(\infty, 2)$ Nikolskii-type ineq. assump. $\Rightarrow$ discretization of $L^p$-norm with $m = CM^2N[\log(4M^2N)]^2$ points $(C = C(\varepsilon, p))$.

# Previous known result for $p < 2$

**Theorem (F. Dai, A. Prymak, A. Shadrin, V. Temlyakov, S. Tikhonov, 20).**
$p \in [1, 2)$, $L$ satisfies $(\infty, 2)$ Nikolskii-type ineq. assump. with $M \leq N^r \Rightarrow$ discretization of $L^p$-norm with $m = CM^2N[\log N]^3$ points ($C = C(r, \varepsilon, p)$).

# The main result for sampling discretization

**Theorem (E.K., 20).** Let
$M \geq 1, p \in (1,\infty), \varepsilon \in (0,1) \Rightarrow$
$\exists C := C(M,p,\varepsilon)$: $\forall$ $N$-dimensional $L$, such that

$$\|f\|_\infty \leq M N^{1/\max\{p,2\}} \|f\|_{\max\{p,2\}} \quad \forall f \in L,$$

$\forall m > CN[\log N]^{\max\{p,2\}}$ there are points $X_1, \dots, X_m$ such that

$$(1-\varepsilon)\|f\|_p^p \leq \frac{1}{m}\sum_{j=1}^m |f(X_j)|^p \leq (1+\varepsilon)\|f\|_p^p \quad \forall\, f \in L.$$

# Improved Guédon–Rudelson bound

**Theorem(E.K., 21).** If $B \subset L$ is $\theta$-convex and $B \subset D$ — Euclidean ball, then for $p \in [\theta, \infty)$ one has

$$\mathbb{E}\big[V_p(B)\big] \leq C\big(A + A^{1/2}(\sup_{f\in B}\mathbb{E}|f(X_1)|^p)^{1/2}\big),$$

$$A = \tfrac{1}{m}\mathbb{E}\big(\sup_{f\in D}\max_{1\leq j\leq m}|f(X_j)|^2 \sup_{h\in B}\max_{1\leq j\leq m}|h(X_j)|^{p-2}\big) + \tfrac{\log m}{m}\mathbb{E}\big(\sup_{h\in B}\max_{1\leq j\leq m}|h(X_j)|^p\big).$$

• $L$ satisfies $(\infty, 2)$ Nikolskii-type inequality $\Rightarrow$ $A \leq 2M^p N^{p/2} \frac{\log m}{m} \Rightarrow$ discretization of $L^p$-norm with $m = CN^{p/2} \log N$ points ($C = C(M, \varepsilon, p)$);

• discretization + Lewis' change of density theorem $\Rightarrow$ J. Bourgain, J. Lindenstrauss, V. Milman result on embeddings of $N$-dimensional $L \subset L^p[0,1]$ into $\ell_p^m$:

$\forall$ $N$-dimensional $L \subset L^p[0,1]$ $\exists$ $N$-dimensional $L' \subset \ell_p^m$, with $m = c_{\varepsilon,p} N^{p/2} \log N$, at a Banach-Mazur distance $\leq 1 + \varepsilon$ from $L$.

• Extensively studied: G. Schechtman, M. Talagrand, J. Bourgain, V. Milman, J. Lindenstrauss, A. Zvavitch...

## The symmetrization argument

Let $R_p(f) = \sum_{j=1}^{m} |f(X_j)|^p$.

**Lemma.**
Assume $\exists r \in (0,1)$: $\forall\ X := \{X_1, \ldots, X_m\}$:

$$\mathbb{E}_\varepsilon \sup_{f \in B} \left| \sum_{j=1}^{m} \varepsilon_j |f(X_j)|^p \right| \le \Theta(X) \sup_{f \in B} \big(R_p(f)\big)^{1-r},$$

$\varepsilon_1, \ldots, \varepsilon_m$ — i.i.d $\pm 1$ symmetric Bernoulli r. v. Then

$$\mathbb{E}\big[V_p(B)\big] \le C(r)\Big[A + A^r \big(\sup_{f \in B} \mathbb{E}|f(X_1)|^p\big)^{1-r}\Big],$$

where $A = \frac{\mathbb{E}[\Theta(X)^{1/r}]}{m}$.

# Generic Chaining

To bound $\mathbb{E}_\varepsilon \sup_{f \in B} \left| \sum_{j=1}^{m} \varepsilon_j |f(X_j)|^p \right|$ we use M. Talagrand's generic chaining.

**Definition.** Let $f \in (F, \varrho)$ be a metric space, $\tau > 0$. Set

$$\gamma_{\tau,1}(F, \varrho) := \inf \sup_{f \in F} \sum_{k=0}^{\infty} 2^{k/\tau} \inf_{g \in F_n} \varrho(f, g),$$

where the infimum is taken over all sequences of sets $F_n$ of cardinality $|F_n| \leq 2^{2^n}$.

# Talagrand's theorem

- Let $\varepsilon_f$ be a random process, $f \in (F, \varrho)$.

Assume that there are numbers $K > 0$ and $\alpha > 0$:

$$P(|\varepsilon_f - \varepsilon_g| \geq K\varrho(f,g)t^{1/\tau}) \leq 2e^{-t} \quad \forall t > 0.$$

Then $\forall\ f_0 \in F$ one has

$$\mathbb{E} \sup_{f \in F} |\varepsilon_f - \varepsilon_{f_0}| \leq C(K,\alpha)\gamma_{\tau,1}(F,\varrho).$$

- In our case $\varepsilon_f = \sum_{j=1}^{m} \varepsilon_j |f(X_j)|^p, \ f \in B.$

# Tails estimate

**Theorem.** $\forall \tau \in [2,\infty)\ \exists C_\tau > 0$:

$$P\Big(\big|\sum_{j=1}^{m} \varepsilon_j c_j\big| \geq C_\tau \big(\sum_{j=1}^{m} |c_j|^{\tau'}\big)^{1/\tau'} t^{1/\alpha}\Big) \leq 2e^{-t},$$

where $\tau' = \frac{\tau}{\tau-1}$.

Thus, in our case

$$\varrho(f,g) = \Big(\sum_{j=1}^{m} \big(|f(X_j)|^p - |g(X_j)|^p\big)^{\tau'}\Big)^{1/\tau'}$$

# Dudley's bound

- Assumptions: $e_k(B, \|\cdot\|_{\infty,X}) \le W_B(X) 2^{-k/\alpha}$.
- We take $\tau := \max\{\alpha, 2\}$ and note that

$$\varrho(f,g) \le C_1 \sup_{h\in B} \|h\|_{\infty,X}^{\frac{p-\alpha}{\tau}} \sup_{v\in B} \big(R_p(v)\big)^{1-\frac{1}{\tau}} \|f-g\|_{\infty,X}^{\alpha/\tau},$$

where $R_p(v) = \sum_{j=1}^{m} |v(X_j)|^p$.

- By the Dudley's entropy bound

$$\gamma_{\tau,1}(B,\varrho) \le C_2 \sum_{k=0}^{\infty} 2^{k/\tau} e_k(B,\varrho) \le$$

$$C_3 \sup_{h\in B} \|h\|_{\infty,X}^{\frac{p-\alpha}{\tau}} \sup_{v\in B} \big(R_p(v)\big)^{1-\frac{1}{\tau}} W_B(X)^{\alpha/\tau} \log m$$

- Using the symmetrization argument, we get

$$\mathbb{E}\big[V_p(B)\big] \leq C\Big(A + A^{\frac{1}{\max\{\alpha,2\}}}\big(\sup_{f\in B}\mathbb{E}|f(X_1)|^p\big)^{1-\frac{1}{\max\{\alpha,2\}}}\Big),$$

where

$$A = \frac{[\log m]^{\max\{\alpha,2\}}}{m}\mathbb{E}\Big([W_B(X)]^\alpha \sup_{f\in B}\max_{1\leq j\leq m}|f(X_j)|^{p-\alpha}\Big).$$

- To improve the power of $\log m$ we have used the recent results on generic chaining of R. van Handel (2018).

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Thank You!